Superluminal Group Velocity of Neutrinos : Review, Development and Problems

Kazuyuki FUJII *

International College of Arts and Sciences†

Yokohama City University

Yokohama, 236–0027

Japan

Abstract

The purpose of this paper is both to provide mathematical reinforcements to the paper [Mecozzi and Bellini : arXiv:1110.1253 [hep-ph]] by taking decoherence into consideration and to present some important problems related.

We claim that neutrinos have superluminality as a latent possibility.

1 Introduction

In September 2011 we encountered a remarkable and unbelievable paper by the OPERA collaboration [1] that **the speed of neutrino exceeds that of light in vacuum**. They measured a collective speed of mu–neutrino flying from CERN to Gran Sasso Laboratory (see for example the Fig. 5 in [1]). However, this result conflicts with special relativity in the most basic sense.

*E-mail address : fujii@yokohama-cu.ac.jp

†New affiliation after the coming April

After the paper appeared in the arXiv researchers in the world published or are preparing many papers on this topic. Some of them agree with the result, while others don't agree according to each author's conviction of the special relativity. See the hep-ph in the arXiv.

We cannot conclude whether this experiment (detection) is correct or not at the present time. Of course, it must be checked by other experiment teams.

Some researchers protested against the OPERA detection strongly. One of main reasons is due to the Kamiokande detection in 1987 [2]. It detected lights and neutrinos coming from the Supernova SN1987A at almost the same time. If the speed of neutrino is faster than that of light in vacuum, it must have detected neutrinos several years earlier.

By the way, Mecozzi and Bellini in [3][1] gave a smart interpretation of the result. They suggested that the result is due to **the superluminal group velocity** of neutrinos arising from superposition (namely, the neutrino mixing) in Quantum Mechanics [4]. The neutrino mixing which is well–known in particle physics is just a quantum mechanical phenomenon.

However, coherence in Quantum Mechanics is affected by environments and it could be destroyed in short time. A long–distance flight from SN1987A might have destroyed coherence of neutrinos, and as a result the superluminal group velocity was lost. The paper [3] offers an interpretation that the OPERA result does not conflict with Kamiokande detection.

In this note we provide mathematical reinforcements to the paper [3] in terms of **decoherence** and would like to offer a "super–smart" interpretation to the OPERA result [5].

2 Superluminal Group Velocity of Neutrinos

In this section we review the paper [3] in detail (because it is a bit unclear from the mathematical point of view).

First, we prepare some notation for convenience. Since we treat a two level system in

[1] K.F gave a small contribution to this paper, see Acknowledgments of the paper.

the following the target space is $\mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}(|\phi_1\rangle, |\phi_2\rangle)$ with bases

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then Pauli matrices $\{\sigma_x,\ \sigma_y,\ \sigma_z\}$ with the identity 1_2

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

act on the space.

The three generations of leptons are

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}.$$

However, each neutrino is not single but slightly mixed like

$$\nu'_\mu = \cos\Theta\ \nu_\mu - \sin\Theta\ \nu_\tau, \quad \nu'_\tau = \sin\Theta\ \nu_\mu + \cos\Theta\ \nu_\tau.$$

This Θ is called the mixing angle in vacuum, which is small enough. Therefore real generations are for example

$$\begin{pmatrix} \mu \\ \nu'_\mu \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \nu'_\tau \end{pmatrix}.$$

Note that the mixing matrix

$$R(\Theta) = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \tag{1}$$

is called the Pontecorvo-Maki-Nakagawa-Sakata matrix ([6], [7], [8]).

Let us start with a model [9], [10]. Since we treat two neutrinos $\{\nu_\mu,\ \nu_\tau\}$ in the paper, the Dirac equation for the two neutrinos can be reduced to a Schrödinger form written in terms of a two component vector of positive energy probability amplitude **in the ultra-relativistic limit**.

Then the two neutrino system can be mapped to a two–level quantum system with distinct energy eigenvalues along with **the assumption of equal fixed momenta** [9].

We set that m_i $(i=1,2)$ are the neutrino masses and E_i are the energies given by the approximation

$$E_i=\sqrt{(cp)^2+(m_ic^2)^2}\approx cp+\frac{m_i^2c^3}{2p}=cp+\frac{m_i^2c^4}{2pc}$$

($p_1=p_2=p$ from the assumption) and Θ is the mixing angle. The Hamiltonian becomes

$$H=H(p)=\left(cp+\frac{\epsilon_0}{2}\right)1_2+\frac{\epsilon}{2}\{\sin(2\Theta)\sigma_x-\cos(2\Theta)\sigma_z\} \tag{2}$$

where

$$\epsilon_0=\frac{(m_1^2+m_2^2)c^4}{2pc},\quad \epsilon=\frac{(m_2^2-m_1^2)c^4}{2pc},\quad E_0=cp+\frac{\epsilon_0}{2}.$$

Here, note that $E_0-\frac{\epsilon}{2}=E_1$ and $E_0+\frac{\epsilon}{2}=E_2$.

Let us rewrite H in (2) in a familiar form by making use of Pauli matrices above

$$H=\begin{pmatrix} cp+\frac{\epsilon_0}{2}-\frac{\epsilon}{2}\cos(2\Theta) & \frac{\epsilon}{2}\sin(2\Theta) \\ \frac{\epsilon}{2}\sin(2\Theta) & cp+\frac{\epsilon_0}{2}+\frac{\epsilon}{2}\cos(2\Theta) \end{pmatrix}. \tag{3}$$

For this it is easy to obtain the eigenvalues

$$E_\pm=cp+\frac{\epsilon_0}{2}\pm\frac{\epsilon}{2}\equiv E_0\pm\frac{\epsilon}{2} \tag{4}$$

and the corresponding eigenvectors

$$\begin{aligned}|\phi_-\rangle &= \begin{pmatrix}\cos\Theta \\ -\sin\Theta\end{pmatrix}=\cos\Theta|\phi_1\rangle-\sin\Theta|\phi_2\rangle, \\ |\phi_+\rangle &= \begin{pmatrix}\sin\Theta \\ \cos\Theta\end{pmatrix}=\sin\Theta|\phi_1\rangle+\cos\Theta|\phi_2\rangle.\end{aligned}$$

If we define a unitary matrix

$$(|\phi_-\rangle,|\phi_+\rangle)=\begin{pmatrix}\cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta\end{pmatrix}=R(\Theta)$$

then we can make H diagonal like

$$H=R(\Theta)\begin{pmatrix}E_0-\frac{\epsilon}{2} & 0 \\ 0 & E_0+\frac{\epsilon}{2}\end{pmatrix}R(\Theta)^\dagger=R(\Theta)\begin{pmatrix}E_1 & 0 \\ 0 & E_2\end{pmatrix}R(\Theta)^T \tag{5}$$

or in a spectral decomposition form

$$H = \left(E_0 - \frac{\epsilon}{2}\right)|\phi_-\rangle\langle\phi_-| + \left(E_0 + \frac{\epsilon}{2}\right)|\phi_+\rangle\langle\phi_+|. \tag{6}$$

These forms are used in the next section.

Next, we label the Hamiltonian by the momentum p. Namely,

$$\mathbf{H} = \mathbf{H}(p) = H(p) \otimes |p\rangle\langle p|. \tag{7}$$

This is in a certain sense the graph of a function.

If we define eigenvectors of $\mathbf{H}$ in (7) as simultaneous ones of both flavor and momentum

$$H \otimes 1|\phi_\pm, p\rangle = E_\pm|\phi_\pm, p\rangle, \quad 1 \otimes \hat{p}|\phi_\pm, p\rangle = p|\phi_\pm, p\rangle,$$

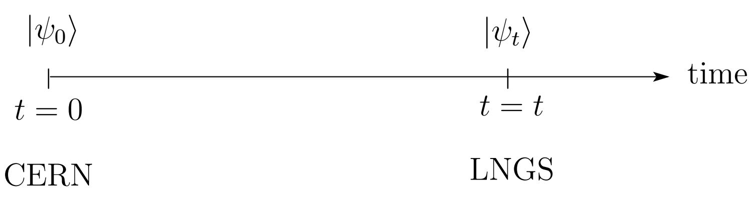
then we have

$$\begin{aligned} |\phi_-, p\rangle &= \begin{pmatrix} \cos\Theta \\ -\sin\Theta \end{pmatrix} \otimes |p\rangle = \cos\Theta|\phi_1, p\rangle - \sin\Theta|\phi_2, p\rangle, \\ |\phi_+, p\rangle &= \begin{pmatrix} \sin\Theta \\ \cos\Theta \end{pmatrix} \otimes |p\rangle = \sin\Theta|\phi_1, p\rangle + \cos\Theta|\phi_2, p\rangle. \end{aligned} \tag{8}$$

Therefore, the spectral decomposition of $\mathbf{H}$ is given by

$$\mathbf{H} = \left(E_0 - \frac{\epsilon}{2}\right)|\phi_-, p\rangle\langle\phi_-, p| + \left(E_0 + \frac{\epsilon}{2}\right)|\phi_+, p\rangle\langle\phi_+, p|. \tag{9}$$

Next, we consider a time–evolution of the system.



The state $|\psi_t\rangle$ at time t is given by

$$|\psi_t\rangle = \int dp' e^{-it\mathbf{H}(p')/\hbar}\,|\psi_0\rangle \tag{10}$$

with the initial state $|\psi_0\rangle$, and straightforward calculation with (9) gives

$$|\psi_t\rangle = \int dp' e^{-itE_0/\hbar} \left\{ e^{-it\frac{\epsilon}{2\hbar}} \langle\phi_+, p'|\psi_0\rangle |\phi_+, p'\rangle + e^{it\frac{\epsilon}{2\hbar}} \langle\phi_-, p'|\psi_0\rangle |\phi_-, p'\rangle \right\} \tag{11}$$

where $E_0 = E_0(p'),\ \epsilon = \epsilon(p'),\ \epsilon_0 = \epsilon_0(p')$ for simplicity.

From now on we assume some conditions :

(a) we start with one neutrino flavor, namely, $\langle\phi_2, p|\psi_0\rangle = 0$,

(b) the initial amplitude of the neutrino waveform is $\langle\phi_1, p|\psi_0\rangle = \langle p|\psi_0\rangle$.

These assumptions seem to be natural.

Then it is easy to see that (11) can be rewritten as

$$\begin{aligned} &|\psi_t\rangle = \int dp' e^{-itE_0/\hbar} \langle p'|\psi_0\rangle \times \\ &\left[e^{-it\frac{\epsilon}{2\hbar}} \left\{ \sin^2\Theta |\phi_1, p'\rangle + \sin\Theta\cos\Theta |\phi_2, p'\rangle \right\} + e^{it\frac{\epsilon}{2\hbar}} \left\{ \cos^2\Theta |\phi_1, p'\rangle - \sin\Theta\cos\Theta |\phi_2, p'\rangle \right\} \right] \end{aligned} \tag{12}$$

by use of (8).

Now let us start detection of neutrino :

(a) first, we perform a flavor measurement (for example, flavor 1),

(b) immediately after (a), we perform a position measurement.

When flavor 1 is detected the collapsed state becomes

$$|\psi_t\rangle \longrightarrow |\psi_t'\rangle = \frac{1}{\sqrt{\mathcal{D}}} \int dp' |\phi_1, p'\rangle\langle\phi_1, p'|\psi_t\rangle = \frac{1}{\sqrt{\mathcal{D}}} \mathcal{P} |\psi_t\rangle \tag{13}$$

where $\mathcal{D}$ is the normalization factor given by

$$\mathcal{D} = \int dp' \langle\psi_t|\phi_1, p'\rangle\langle\phi_1, p'|\psi_t\rangle = \langle\psi_t|\mathcal{P}|\psi_t\rangle \tag{14}$$

($\langle\psi_t'|\psi_t'\rangle = 1$) and

$$\mathcal{P} \equiv \int dp' |\phi_1, p'\rangle\langle\phi_1, p'|$$

is the projection operator to the flavor 1 state (: $\mathcal{P}^2 = \mathcal{P},\ \mathcal{P}^\dagger = \mathcal{P}$).

It is in general difficult to perform a position measurement immediately after flavor 1 is detected, so we average the positions of neutrinos. We believe that this replacement is not so bad.

The expectation value of position measurement on the collapsed state $|\psi_t'\rangle$ is given by

$$\langle x_t\rangle = \int dp' \langle\psi_t'|p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle p'|\psi_t'\rangle.$$

This is a kind of definition. Note that

$$i\hbar\frac{\partial}{\partial p}$$

is a position operator, because

$$i\hbar\frac{\partial}{\partial p}\langle p|x\rangle = i\hbar\frac{\partial}{\partial p}e^{-ixp/\hbar} = xe^{-ixp/\hbar} = x\langle p|x\rangle.$$

We must evaluate the expectation value of position $\langle x_t\rangle$. From (13) and noting the formula

$$\langle\phi_1, p'|\phi_1, p''\rangle = \delta(p' - p'') \tag{15}$$

we have and set

$$\langle x_t\rangle = \frac{1}{\mathcal{D}}\int dp' \langle\psi_t|\phi_1, p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle\phi_1, p'|\psi_t\rangle \equiv \frac{\mathcal{N}}{\mathcal{D}} \tag{16}$$

for simplicity.

First, let us calculate $\langle\phi_1, p'|\psi_t\rangle$. From (12) and the formula (15) it is easy to see

$$\langle\phi_1, p'|\psi_t\rangle = \langle p'|\psi_0\rangle F(p') \tag{17}$$

where

$$F(p') = e^{-itE_0/\hbar}\left(\sin^2\Theta e^{-it\epsilon/2\hbar} + \cos^2\Theta e^{it\epsilon/2\hbar}\right). \tag{18}$$

Then the normalization factor $\mathcal{D}$ in (14) becomes

$$\mathcal{D} = \int dp' |\langle p'|\psi_0\rangle|^2 |F(p')|^2.$$

Now, we make another assumption. The initial distribution of the neutrino momentum $|\langle p'|\psi_0\rangle|^2$ is narrow with respect to $F(p')$, and centered on $p' = p$ (p is fixed). Namely,

$$|\langle p'|\psi_0\rangle|^2 \approx \delta(p' - p). \tag{19}$$

We believe this one natural. Then

$$\mathcal{D} \approx |F(p)|^2. \tag{20}$$

For later convenience we calculate (20). Here is an elementary formula

Formula for $\alpha,\ \beta \in \mathbf{R}$

$$|\alpha e^{-i\theta} + \beta e^{i\theta}|^2 = (\alpha+\beta)^2 - 4\alpha\beta \sin^2\theta.$$

This gives

$$\begin{aligned}
|F(p)|^2 &= |\sin^2\Theta e^{-it\epsilon/2\hbar} + \cos^2\Theta e^{it\epsilon/2\hbar}|^2 \\
&= \left(\sin^2\Theta + \cos^2\Theta\right)^2 - 4\sin^2\Theta\cos^2\Theta\sin^2\left(\frac{t\epsilon}{2\hbar}\right) \\
&= 1 - (2\sin\Theta\cos\Theta)^2\sin^2\left(\frac{t\epsilon}{2\hbar}\right) \\
&= 1 - \sin^2(2\Theta)\sin^2\left(\frac{t\epsilon}{2\hbar}\right) \qquad (21)
\end{aligned}$$

by use of (18) (note that $\epsilon = \epsilon(p)$).

By inserting the equation (17) into $\mathcal{N}$ in (16) we have and set

$$\begin{aligned}
\mathcal{N} &= \int dp' \langle\psi_t|\phi_1, p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle\phi_1, p'|\psi_t\rangle \\
&= \int dp' \bar{F}(p')\langle\psi_0|p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\{\langle p'|\psi_0\rangle F(p')\} \\
&= \int dp' |F(p')|^2\langle\psi_0|p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle p'|\psi_0\rangle + \int dp' |\langle p'|\psi_0\rangle|^2 \bar{F}(p')\left(i\hbar\frac{\partial}{\partial p'}\right)F(p') \\
&\equiv \mathcal{N}_1 + \mathcal{N}_2
\end{aligned}$$

for simplicity. Next, let us calculate $\mathcal{N}_1$ and $\mathcal{N}_2$ separately.

From

$$\mathcal{N}_1 = \mathrm{Re}\mathcal{N}_1 + i\mathrm{Im}\mathcal{N}_1$$

we have

$$\begin{aligned}
\mathrm{Im}\mathcal{N}_1 &= \frac{1}{2i}\left(\mathcal{N}_1 - \bar{\mathcal{N}}_1\right) \\
&= \frac{\hbar}{2}\int dp'|F(p')|^2 \left\{ \langle\psi_0|p'\rangle \frac{\partial}{\partial p'}\langle p'|\psi_0\rangle + \langle p'|\psi_0\rangle \frac{\partial}{\partial p'}\langle\psi_0|p'\rangle \right\} \\
&= \frac{\hbar}{2}\int dp'|F(p')|^2 \frac{\partial}{\partial p'}\left\{\langle p'|\psi_0\rangle\langle\psi_0|p'\rangle\right\} \\
&= \frac{\hbar}{2}\int dp'|F(p')|^2 \frac{\partial}{\partial p'}|\langle p'|\psi_0\rangle|^2 \\
&\quad \text{(integration by parts)} \\
&= -\frac{\hbar}{2}\int dp'|\langle p'|\psi_0\rangle|^2 \frac{\partial}{\partial p'}|F(p')|^2 \\
&\approx -\frac{\hbar}{2}\int dp'\delta(p'-p)\frac{\partial}{\partial p'}|F(p')|^2 = -\frac{\hbar}{2}\frac{\partial}{\partial p}|F(p)|^2 \qquad (22)
\end{aligned}$$

by use of the assumption in (19). Similarly, we have

$$\begin{aligned}
\mathrm{Re}\mathcal{N}_1 &= \frac{1}{2}\left(\mathcal{N}_1 + \bar{\mathcal{N}}_1\right) \\
&= \frac{1}{2}\int dp'|F(p')|^2 \left\{ \langle\psi_0|p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle p'|\psi_0\rangle - \langle p'|\psi_0\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle\psi_0|p'\rangle \right\}.
\end{aligned}$$

The range of integration is narrow enough because of the assumption in (19), so we approximate the integration like

$$\begin{aligned}
\mathrm{Re}\mathcal{N}_1 &\approx |F(p)|^2\,\frac{1}{2}\int dp' \left\{ \langle\psi_0|p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle p'|\psi_0\rangle - \langle p'|\psi_0\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle\psi_0|p'\rangle \right\} \\
&\quad \text{(integration by parts)} \\
&= |F(p)|^2\int dp' \left\{ \langle\psi_0|p'\rangle \left(i\hbar\frac{\partial}{\partial p'}\right)\langle p'|\psi_0\rangle \right\} \\
&= |F(p)|^2\,\langle x_0\rangle. \qquad (23)
\end{aligned}$$

Therefore, we obtain the approximate value

$$\mathcal{N}_1 = \mathrm{Re}\mathcal{N}_1 + i\mathrm{Im}\mathcal{N}_1 \approx |F(p)|^2\langle x_0\rangle - \frac{i\hbar}{2}\frac{\partial}{\partial p}|F(p)|^2. \qquad (24)$$

For $\mathcal{N}_2$ we have

$$\begin{aligned}
\mathcal{N}_2 &= \int dp'|\langle p'|\psi_0\rangle|^2 \bar{F}(p')\left(i\hbar\frac{\partial}{\partial p'}\right)F(p') \\
&\approx \bar{F}(p)\left(i\hbar\frac{\partial}{\partial p}\right)F(p) = i\hbar\bar{F}(p)\frac{\partial}{\partial p}F(p) \qquad (25)
\end{aligned}$$

by use of the assumption in (19).

Then (24) and (25) give

$$\begin{aligned}\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2 &= |F(p)|^2\langle x_0\rangle-\frac{i\hbar}{2}\frac{\partial}{\partial p}|F(p)|^2+i\hbar\bar{F}(p)\frac{\partial}{\partial p}F(p)\\ &= |F(p)|^2\langle x_0\rangle+\frac{i\hbar}{2}\left\{\bar{F}(p)\frac{\partial}{\partial p}F(p)-\frac{\partial}{\partial p}\bar{F}(p)F(p)\right\}\end{aligned}\tag{26}$$

and (16) and (20) give (the approximate value)

$$\langle x_t\rangle=\frac{\mathcal{N}}{\mathcal{D}}=\frac{\mathcal{N}}{|F(p)|^2}=\langle x_0\rangle+\frac{i\hbar}{2}\frac{\bar{F}(p)\frac{\partial}{\partial p}F(p)-\frac{\partial}{\partial p}\bar{F}(p)F(p)}{|F(p)|^2}.\tag{27}$$

Next, let us calculate the right hand side of (27) by use of (18) :

$$F(p)=e^{-itE_0/\hbar}\left(\sin^2\Theta e^{-it\epsilon/2\hbar}+\cos^2\Theta e^{it\epsilon/2\hbar}\right).$$

Noting $E_0=E_0(p),\ \epsilon=\epsilon(p)$ we have

$$\frac{\partial}{\partial p}F(p)=-i\frac{t}{\hbar}\frac{\partial E_0}{\partial p}F(p)-i\frac{t}{2\hbar}\frac{\partial\epsilon}{\partial p}e^{-itE_0/\hbar}\left(\sin^2\Theta e^{-it\epsilon/2\hbar}-\cos^2\Theta e^{it\epsilon/2\hbar}\right)$$

and

$$\begin{aligned}\bar{F}(p)\frac{\partial}{\partial p}F(p) &= -i\frac{t}{\hbar}\frac{\partial E_0}{\partial p}|F(p)|^2\\ &\quad -i\frac{t}{2\hbar}\frac{\partial\epsilon}{\partial p}\left(\sin^2\Theta e^{-it\epsilon/2\hbar}-\cos^2\Theta e^{it\epsilon/2\hbar}\right)\left(\sin^2\Theta e^{it\epsilon/2\hbar}+\cos^2\Theta e^{-it\epsilon/2\hbar}\right)\\ &= -i\frac{t}{\hbar}\frac{\partial E_0}{\partial p}|F(p)|^2-i\frac{t}{2\hbar}\frac{\partial\epsilon}{\partial p}\left(\sin^4\Theta-\cos^4\Theta+**\right)\\ &= -i\frac{t}{\hbar}\frac{\partial E_0}{\partial p}|F(p)|^2-i\frac{t}{2\hbar}\frac{\partial\epsilon}{\partial p}\left(\sin^2\Theta-\cos^2\Theta+**\right)\end{aligned}$$

where $**$ is the terms which will be neglected at the final stage. This gives

$$\begin{aligned}\frac{i\hbar}{2}\left(\bar{F}(p)\frac{\partial}{\partial p}F(p)-\text{c.c.}\right) &= t\frac{\partial E_0}{\partial p}|F(p)|^2+\frac{t}{2}\frac{\partial\epsilon}{\partial p}\left(\sin^2\Theta-\cos^2\Theta\right)\\ &= t\frac{\partial E_0}{\partial p}|F(p)|^2-\frac{t}{2}\frac{\partial\epsilon}{\partial p}\cos(2\Theta)\end{aligned}$$

and $\langle x_t\rangle$ in (27) is given by

$$\langle x_t\rangle=\langle x_0\rangle+t\frac{\partial E_0}{\partial p}-\frac{t}{2}\frac{\cos(2\Theta)}{|F(p)|^2}\frac{\partial\epsilon}{\partial p}.\tag{28}$$

Definition The group velocity v_g is given by

$$\langle x_t \rangle - \langle x_0 \rangle = v_g t \Longleftrightarrow v_g = \frac{\langle x_t \rangle - \langle x_0 \rangle}{t}.$$

Therefore it becomes

$$v_g = \frac{\partial E_0}{\partial p} - \frac{1}{2} \frac{\cos(2\Theta)}{|F(p)|^2} \frac{\partial \epsilon}{\partial p} \tag{29}$$

from the result above.

Remembering

$$E_0 = cp + \frac{\epsilon_0}{2}, \quad \epsilon_0 = \frac{(m_1^2 + m_2^2)c^4}{2pc}, \quad \epsilon = \frac{(m_2^2 - m_1^2)c^4}{2pc}$$

from (2) simple calculation gives

$$\frac{\partial E_0}{\partial p} = c - \frac{\epsilon_0}{2p}, \quad \frac{\partial \epsilon}{\partial p} = -\frac{\epsilon}{p}$$

and by inserting the above into (29) we have

$$v_g = c - \frac{\epsilon_0}{2p} + \mathcal{S} \frac{\epsilon}{2p} \tag{30}$$

where $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{\cos(2\Theta)}{|F(p)|^2} = \frac{\cos(2\Theta)}{1 - \sin^2(2\Theta) \sin^2\left(\frac{t\epsilon}{2\hbar}\right)} \tag{31}$$

from (21).

As a result we obtain

Theorem (Mecozzi and Bellini) The group velocity v_g is given by

$$v_g = c - \frac{\epsilon_0}{2p} + \frac{\cos(2\Theta)}{1 - \sin^2(2\Theta) \sin^2\left(\frac{t\epsilon}{2\hbar}\right)} \frac{\epsilon}{2p}. \tag{32}$$

Note that the term $\mathcal{S}$ was obtained from a quantum effect (the neutrino mixing) and this plays a definite role in **Superluminal Group Velocity**.

Let us analyze the theorem. For the purpose we set

$$\alpha \equiv \sin^2\left(\frac{t\epsilon}{2\hbar}\right)$$

to look for some condition satisfying $\mathcal{S} > 1$. Namely,

$$\begin{aligned}
& \mathcal{S} = \frac{\cos(2\Theta)}{1 - \alpha\sin^2(2\Theta)} > 1 \\
\Longleftrightarrow \quad & \cos(2\Theta) > 1 - \alpha\sin^2(2\Theta) = 1 - \alpha(1 - \cos^2(2\Theta)) \\
\Longleftrightarrow \quad & (1 - \cos^2(2\Theta))\alpha > 1 - \cos(2\Theta) \quad (\text{excepting } 1 = \cos(2\Theta)) \\
\Longleftrightarrow \quad & \alpha > \frac{1}{1 + \cos(2\Theta)} > \frac{1}{2}.
\end{aligned}$$

Therefore, we obtain

$$\alpha = \sin^2\left(\frac{t\epsilon}{2\hbar}\right) > \frac{1}{2} \implies \mathcal{S} > 1. \tag{33}$$

Here, we assume $m_2 \gg m_1$. Then $\epsilon \approx \epsilon_0$ from (2) and $\mathcal{S} > 1$ gives

$$v_g \approx c + (\mathcal{S} - 1)\frac{\epsilon_0}{2p} > c.$$

As a result

Corollary 1 Under the conditions $\alpha > \frac{1}{2}$ and $m_2 \gg m_1$ we have the superluminal group velocity

$$v_g > c. \tag{34}$$

Note In general, the higher the generation, the heavier corresponding mass. Therefore, the assumption $m_2 \gg m_1$ is not unnatural.

If $\Theta = 0$ (no neutrino mixing) then $\mathcal{S} = 1$ from (31) and we have

$$v_g \approx c - \frac{\epsilon_0}{2p} + \frac{\epsilon_0}{2p} = c$$

under $m_2 \gg m_1$. As a result

Corollary 2 If $\Theta = 0$ we have the usual velocity

$$v_g = c \tag{35}$$

under $\alpha > \frac{1}{2}$ and $m_2 \gg m_1$.

We cannot help admitting a mechanism which accelerates neutrinos arising from the neutrino oscillation. How do we interpret the result ? What is the relation to special relativity ? In the last part of the paper [3] they write :

"Of course, this does not mean that the speed of a possible signal transmitted with a neutrino wave-packet exceeds the speed of light, it is just a property that comes from the wave-packet deformation caused by the interference of the two possible quantum paths that a neutrino may follow before reaching the detector".

Unfortunately, the author cannot understand what they meant and therefore present the following

Problem Give a mathematical expression to their claim.

3 Decoherence of the Neutrino Oscillation

In this section, for the model in the previous section we take **decoherence** into consideration in order to make it more realistic. Then we can build a bridge between Corollary 1 and Corollary 2. For a general introduction to this topic see for example [11].

First of all we present the following

Problem Is there no problem to apply theory of decoherence to neutrinos in a long–distance flight ?

Although this problem is very subtle, let us proceed to the discussion of decoherence.

Since the two neutrino system can be mapped to the two level system we prepare some notation from Quantum Optics. For

$$\sigma_+ \equiv \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

it is easy to see

$$\sigma_+\sigma_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_-\sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us remember

$$H = R(\Theta)\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}R(\Theta)^T$$
$$= \begin{pmatrix} E_1\cos^2\Theta + E_2\sin^2\Theta & (E_2-E_1)\sin\Theta\cos\Theta \\ (E_2-E_1)\sin\Theta\cos\Theta & E_1\sin^2\Theta + E_2\cos^2\Theta \end{pmatrix} \tag{36}$$

from (5) and set

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{37}$$

Note that H and H_0 are symmetric matrices ($H = H^T$, $H_0 = H_0^T$).

To treat decoherence in a correct manner we must change models based on from a pure state to a density matrix. The general definition of density matrix ρ is given by both $\rho^\dagger = \rho$ and $\mathrm{tr}\rho = 1$, so we can write $\rho = \rho(t)$ as

$$\rho = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \quad (a = \bar{a},\ d = \bar{d},\ a + d = 1). \tag{38}$$

The general form of master equation is well–known to be

$$\frac{d}{dt}\rho = -i[H,\rho] + D\rho \quad (\Leftarrow \hbar = 1 \text{ for simplicity}) \tag{39}$$

where

$$D\rho = \mu\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right) + \nu\left(\sigma_+\rho\sigma_- - \frac{1}{2}\sigma_-\sigma_+\rho - \frac{1}{2}\rho\sigma_-\sigma_+\right)$$

and $\mu > \nu > 0$ [2]. We must solve the equation.

If we write H in (36) as

$$H = \begin{pmatrix} h & k \\ k & l \end{pmatrix} \quad (h,\ k,\ l\ \in \mathbf{R}) \tag{40}$$

[2] we don't know how to determine the precise values of μ and ν in our system

for simplicity, then the master equation above can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix} = \begin{pmatrix} -\mu & ik & -ik & \nu \\ ik & i(l-h)-\frac{\mu+\nu}{2} & 0 & -ik \\ -ik & 0 & -i(l-h)-\frac{\mu+\nu}{2} & ik \\ \mu & -ik & ik & -\nu \end{pmatrix}\begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix}. \tag{41}$$

We leave the derivation to readers.

Note and set

$$\begin{pmatrix} -\mu & ik & -ik & \nu \\ ik & i(l-h)-\frac{\mu+\nu}{2} & 0 & -ik \\ -ik & 0 & -i(l-h)-\frac{\mu+\nu}{2} & ik \\ \mu & -ik & ik & -\nu \end{pmatrix}$$

$$= \begin{pmatrix} 0 & ik & -ik & 0 \\ ik & i(l-h) & 0 & -ik \\ -ik & 0 & -i(l-h) & ik \\ 0 & -ik & ik & 0 \end{pmatrix} + \begin{pmatrix} -\mu & 0 & 0 & \nu \\ 0 & -\frac{\mu+\nu}{2} & 0 & 0 \\ 0 & 0 & -\frac{\mu+\nu}{2} & 0 \\ \mu & 0 & 0 & -\nu \end{pmatrix} \equiv \widehat{H} + \widehat{D}.$$

The general solution of (41) is given by

$$\begin{pmatrix} a(t) \\ b(t) \\ \bar{b}(t) \\ d(t) \end{pmatrix} = e^{t\left(\widehat{H}+\widehat{D}\right)}\begin{pmatrix} a(0) \\ b(0) \\ \bar{b}(0) \\ d(0) \end{pmatrix}. \tag{42}$$

However, it is not easy to calculate the term $e^{t\left(\widehat{H}+\widehat{D}\right)}$ exactly, so we use a simple approximation

$$e^{t\left(\widehat{H}+\widehat{D}\right)} = e^{t\left(\widehat{D}+\widehat{H}\right)} \approx e^{t\widehat{D}}e^{t\widehat{H}}.$$

In general, we must use the Zassenhaus formula, see for example [12], [13]. Therefore, we

treat the approximate solution

$$\begin{pmatrix} a(t) \\ b(t) \\ \bar{b}(t) \\ d(t) \end{pmatrix} \approx e^{t\widehat{D}} e^{t\widehat{H}} \begin{pmatrix} a(0) \\ b(0) \\ \bar{b}(0) \\ d(0) \end{pmatrix}. \tag{43}$$

First, we calculate $e^{t\widehat{D}}$. For the purpose we set

$$K = \begin{pmatrix} -\mu & \nu \\ \mu & -\nu \end{pmatrix}$$

and calculate e^{tK}. The eigenvalues of K are $\{0, -(\mu+\nu)\}$ and corresponding eigenvectors (not normalized) are

$$0 \longleftrightarrow \begin{pmatrix} \nu \\ \mu \end{pmatrix}, \quad -(\mu+\nu) \longleftrightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

If we define the matrix

$$O = \begin{pmatrix} \nu & 1 \\ \mu & -1 \end{pmatrix} \Longrightarrow O^{-1} = \frac{1}{\mu+\nu} \begin{pmatrix} 1 & 1 \\ \mu & -\nu \end{pmatrix}$$

then it is easy to see

$$K = O \begin{pmatrix} 0 & \\ & -(\mu+\nu) \end{pmatrix} O^{-1}$$

and

$$e^{tK} = O \begin{pmatrix} 1 & \\ & e^{-t(\mu+\nu)} \end{pmatrix} O^{-1} = \frac{1}{\mu+\nu} \begin{pmatrix} \nu + \mu e^{-t(\mu+\nu)} & \nu - \nu e^{-t(\mu+\nu)} \\ \mu - \mu e^{-t(\mu+\nu)} & \mu + \nu e^{-t(\mu+\nu)} \end{pmatrix}.$$

Therefore, we have

$$e^{t\widehat{D}} = \begin{pmatrix} \frac{\nu+\mu e^{-t(\mu+\nu)}}{\mu+\nu} & 0 & 0 & \frac{\nu-\nu e^{-t(\mu+\nu)}}{\mu+\nu} \\ 0 & e^{-t\frac{\mu+\nu}{2}} & 0 & 0 \\ 0 & 0 & e^{-t\frac{\mu+\nu}{2}} & 0 \\ \frac{\mu-\mu e^{-t(\mu+\nu)}}{\mu+\nu} & 0 & 0 & \frac{\mu+\nu e^{-t(\mu+\nu)}}{\mu+\nu} \end{pmatrix} \approx \frac{1}{\mu+\nu} \begin{pmatrix} \nu & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & \mu \end{pmatrix} \tag{44}$$

if t is large enough ($t \gg 1/\nu$).

Next, we calculate $e^{t\widehat{H}}$. Since we need some properties of tensor product in the following see for example [13]. We can write the equation as

$$\widehat{H} = -i\,(H \otimes 1_2 - 1_2 \otimes H) \quad (\Longleftarrow H = H^T).$$

In fact,

$$\begin{aligned}
\widehat{H} &= -i\left\{\begin{pmatrix} h & k \\ k & l \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} h & k \\ k & l \end{pmatrix}\right\} \\
&= -i\left\{\begin{pmatrix} h & 0 & k & 0 \\ 0 & h & 0 & k \\ k & 0 & l & 0 \\ 0 & k & 0 & l \end{pmatrix} - \begin{pmatrix} h & k & 0 & 0 \\ k & l & 0 & 0 \\ 0 & 0 & h & k \\ 0 & 0 & k & l \end{pmatrix}\right\} = -i\begin{pmatrix} 0 & -k & k & 0 \\ -k & -(l-h) & 0 & k \\ k & 0 & l-h & -k \\ 0 & k & 0-k & 0 \end{pmatrix}.
\end{aligned}$$

It is well–known that

$$e^{t\widehat{H}} = e^{-it(H\otimes 1_2 - 1_2 \otimes H)} = e^{-itH\otimes 1_2} e^{it1_2\otimes H} = \left(e^{-itH} \otimes 1_2\right)\left(1_2 \otimes e^{itH}\right) = e^{-itH} \otimes e^{itH}.$$

Since

$$H = R(\Theta)\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} R(\Theta)^T$$

we have

$$e^{-itH} = R(\Theta)\begin{pmatrix} e^{-itE_1} & 0 \\ 0 & e^{-itE_2} \end{pmatrix} R(\Theta)^T$$

and

$$\begin{aligned}
e^{t\widehat{H}} &= \left\{R(\Theta)\begin{pmatrix} e^{-itE_1} & 0 \\ 0 & e^{-itE_2} \end{pmatrix} R(\Theta)^T\right\} \otimes \left\{R(\Theta)\begin{pmatrix} e^{itE_1} & 0 \\ 0 & e^{itE_2} \end{pmatrix} R(\Theta)^T\right\} \\
&= (R(\Theta) \otimes R(\Theta))\left\{\begin{pmatrix} e^{-itE_1} & 0 \\ 0 & e^{-itE_2} \end{pmatrix} \otimes \begin{pmatrix} e^{itE_1} & 0 \\ 0 & e^{itE_2} \end{pmatrix}\right\} (R(\Theta) \otimes R(\Theta))^T \\
&= (R(\Theta) \otimes R(\Theta))\begin{pmatrix} 1 & & & \\ & e^{it(E_2-E_1)} & & \\ & & e^{-it(E_2-E_1)} & \\ & & & 1 \end{pmatrix} (R(\Theta) \otimes R(\Theta))^T.
\end{aligned}$$

Since

$$R(\Theta) = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

we have

$$R(\Theta)\otimes R(\Theta) = \begin{pmatrix} \cos^2\Theta & \cos\Theta\sin\Theta & \cos\Theta\sin\Theta & \sin^2\Theta \\ -\cos\Theta\sin\Theta & \cos^2\Theta & -\sin^2\Theta & \cos\Theta\sin\Theta \\ -\cos\Theta\sin\Theta & -\sin^2\Theta & \cos^2\Theta & \cos\Theta\sin\Theta \\ \sin^2\Theta & -\cos\Theta\sin\Theta & -\cos\Theta\sin\Theta & \cos^2\Theta \end{pmatrix}$$

and, by setting $J = e^{it(E_2-E_1)} = e^{it\epsilon}$ for simplicity,

$$\begin{aligned} e^{t\widehat{H}} &= R(\Theta)\otimes R(\Theta) \begin{pmatrix} 1 & & & \\ & J & & \\ & & J^{-1} & \\ & & & 1 \end{pmatrix} \left(R(\Theta)\otimes R(\Theta)\right)^T \\ &= \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ * & * & * & * \\ * & * & * & * \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} \alpha_{11} &= \cos^4\Theta + (J+J^{-1})\cos^2\Theta\sin^2\Theta + \sin^4\Theta, \\ \alpha_{12} &= -\cos^3\Theta\sin\Theta + J\cos^3\Theta\sin\Theta - J^{-1}\cos\Theta\sin^3\Theta + \cos\Theta\sin^3\Theta, \\ \alpha_{13} &= -\cos^3\Theta\sin\Theta - J\cos\Theta\sin^3\Theta + J^{-1}\cos^3\Theta\sin\Theta + \cos\Theta\sin^3\Theta, \\ \alpha_{14} &= \cos^2\Theta\sin^2\Theta - (J+J^{-1})\cos^2\Theta\sin^2\Theta + \cos^2\Theta\sin^2\Theta \end{aligned}$$

and

$$\begin{aligned} \alpha_{41} &= \cos^2\Theta\sin^2\Theta - (J+J^{-1})\cos^2\Theta\sin^2\Theta + \cos^2\Theta\sin^2\Theta, \\ \alpha_{42} &= -\cos\Theta\sin^3\Theta - J\cos^3\Theta\sin\Theta + J^{-1}\cos\Theta\sin^3\Theta + \cos^3\Theta\sin\Theta, \\ \alpha_{43} &= -\cos\Theta\sin^3\Theta + J\cos\Theta\sin^3\Theta - J^{-1}\cos^3\Theta\sin\Theta + \cos^3\Theta\sin\Theta, \\ \alpha_{44} &= \sin^4\Theta + (J+J^{-1})\cos^2\Theta\sin^2\Theta + \cos^4\Theta. \end{aligned}$$

Note that $*$'s in the matrix are elements not used in the latter. We leave this derivation to readers.

Here, we list very important relations among $\{\alpha\}$

$$\alpha_{11} + \alpha_{41} = 1, \quad \alpha_{12} + \alpha_{42} = 0, \quad \alpha_{13} + \alpha_{43} = 0, \quad \alpha_{14} + \alpha_{44} = 1. \tag{45}$$

Therefore, from (43) and (45) we obtain

$$\begin{aligned}
\begin{pmatrix} a(t) \\ b(t) \\ \bar{b}(t) \\ d(t) \end{pmatrix} &\approx \frac{1}{\mu+\nu} \begin{pmatrix} \nu & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ * & * & * & * \\ * & * & * & * \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \begin{pmatrix} a(0) \\ b(0) \\ \bar{b}(0) \\ d(0) \end{pmatrix} \\
&= \frac{1}{\mu+\nu} \begin{pmatrix} \nu & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} a(0) \\ b(0) \\ \bar{b}(0) \\ d(0) \end{pmatrix}
\end{aligned} \tag{46}$$

for $t \gg 1/\nu$.

For H_0 in (37)

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

we can perform the same process much easily. The master equation is

$$\frac{d}{dt} \begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix} = \begin{pmatrix} -\mu & 0 & -0 & \nu \\ 0 & i(l-h) - \frac{\mu+\nu}{2} & 0 & 0 \\ 0 & 0 & -i(l-h) - \frac{\mu+\nu}{2} & 0 \\ \mu & 0 & 0 & -\nu \end{pmatrix} \begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix}$$

and the (exact) solution is given by

$$\begin{pmatrix} a(t) \\ b(t) \\ \bar{b}(t) \\ d(t) \end{pmatrix} = \begin{pmatrix} \frac{\nu+\mu e^{-t(\mu+\nu)}}{\mu+\nu} & 0 & 0 & \frac{\nu-\nu e^{-t(\mu+\nu)}}{\mu+\nu} \\ 0 & e^{it(l-h)} e^{-t\frac{\mu+\nu}{2}} & 0 & 0 \\ 0 & 0 & e^{-it(l-h)} e^{-t\frac{\mu+\nu}{2}} & 0 \\ \frac{\mu-\mu e^{-t(\mu+\nu)}}{\mu+\nu} & 0 & 0 & \frac{\mu+\nu e^{-t(\mu+\nu)}}{\mu+\nu} \end{pmatrix} \begin{pmatrix} a(0) \\ b(0) \\ \bar{b}(0) \\ d(0) \end{pmatrix}.$$

When $t \gg 1/\nu$ we obtain

$$\begin{pmatrix} a(t) \\ b(t) \\ \bar{b}(t) \\ d(t) \end{pmatrix} \approx \frac{1}{\mu+\nu} \begin{pmatrix} \nu & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} a(0) \\ b(0) \\ \bar{b}(0) \\ d(0) \end{pmatrix}. \tag{47}$$

As a result we have

Theorem Two systems (master equations) whose Hamiltonians are H and H_0 have the same asymptotic behavior (46) and (47) under our approximation.

This theorem implies

Corollary The mixing angle Θ will become 0 if t is large enough.

From both this corollary and corollary 2 in the preceding section we can conclude that the speed of neutrinos (after a long–distance flight) is just that of light in vacuum.

By the way, our calculation in this section is based on a simple approximation. This is a bit poor, so we present the following

Problem Give the explicit (full) calculation.

As for interesting topics of decoherence (which is essential in Quantum Physics) arising from Quantum Optics or Quantum Computation see our papers [14], [15] and [16], [17].

At the end of this section, one comment is in order. It seems to the author that Neutrino Physics gets along with Quantum Optics or Quantum Computation, see for example [18]. In order to make some (deep) relations clear further studies will be required.

4 Concluding Remarks

In this paper we re-examined the paper [3] by Mecozzi and Bellini in detail and tried to give mathematical reinforcements to it by taking decoherence into consideration. Our conclusion

is

Neutrinos are latently superluminal.

We would like to present the following

Problem Re-check our result from a different point of view.

Let us write once more that our argument is based on **group velocity**. Therefore, it is a bit unsatisfactory.

Whether the OPERA experiment is correct or not is not concluded at the present time and it must be checked by other experiment teams. However, such a check will take time. Therefore, it is very important for us to state

Problem Make some (inside) questions clear from a theoretical point of view.

Regarding papers related to this topic see for example [19].

The work is a great challenge to not only (young) Physicists but also (young) Mathematicians, so we conclude the paper by citing famous sentences by late Steve Jobs [3]

Stay hungry, Stay foolish

(see the Concluding Remarks in [5]).

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